Mechanics

Physics 151

Lecture 18

Hamiltonian Equations of Motion

(Chapter 8)

What's Ahead

- We are starting Hamiltonian formalism
  - Hamiltonian equation – Today and 11/26
  - Canonical transformation – 12/3, 12/5, 12/10
  - Close link to non-relativistic QM
- May not cover Hamilton-Jacobi theory
  - Cute but not very relevant
- What shall we do in the last 2 lectures?
  - Classical chaos?
  - Perturbation theory?
  - Classical field theory?
  - Send me e-mail if you have preference!

Hamiltonian Formalism

- Newtonian → Lagrangian → Hamiltonian
  - Describe same physics and produce same results
  - Difference is in the viewpoints
    - Symmetries and invariance more apparent
    - Flexibility of coordinate transformation
- Hamiltonian formalism linked to the development of
  - Hamilton-Jacobi theory
  - Classical perturbation theory
  - Quantum mechanics
  - Statistical mechanics

Lagrange → Hamilton

- Lagrange's equations for n coordinates

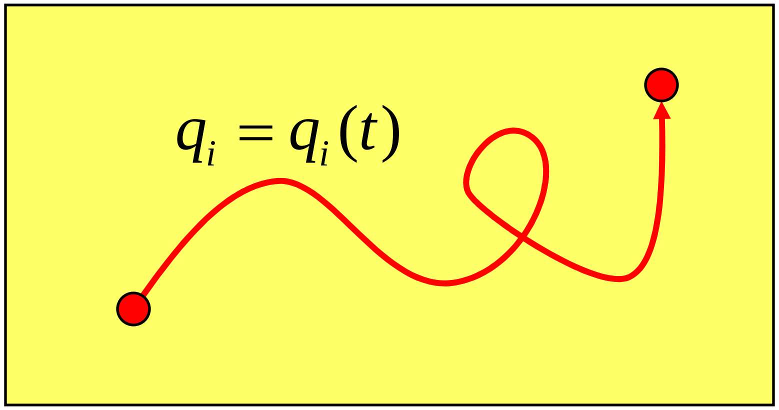
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, n$$

2nd-order differential equation of n variables

  - n equations → $2n$ initial conditions $q_i(t=0)$ $\dot{q}_i(t=0)$
- Can we do with 1st-order differential equations?
  - Yes, but you'll need $2n$ equations
  - We keep q_i and replace $\dot{q}_i$ with something similar
  - We take the conjugate momenta $p_i \equiv \dfrac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}$

Configuration Space

- We considered $(q_1, \ldots, q_n)$ as a point in an n-dim. space
  - Called configuration space
  - Motion of the system → A curve in the config space



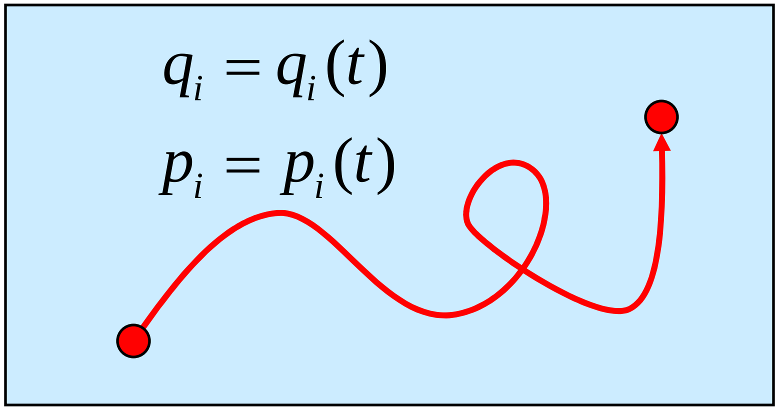
- When we take variations, we consider q_i and $\dot{q}_i$ as independent variables
  - i.e., we have $2n$ independent variables in n-dim. space
  - Isn't it more natural to consider the motion in $2n$-dim space?

Phase Space

- Consider coordinates and momenta as independent
  - State of the system is given by $(q_1, \dots, q_n, p_1, \dots, p_n)$
  - Consider it a point in the $2n$-dimensional phase space
- We are switching the independent variables

  $$(q_i, \dot{q}_i, t) \rightarrow (q_i, p_i, t)$$

  - A bit of mathematical trick is needed to do this



Legendre Transformation

- Start from a function of two variables $f(x, y)$
  - Total derivative is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \equiv udx + vdy$$

  - Define $g \equiv f - ux$ and consider its total derivative

$$dg = df - d(ux) = udx + vdy - udx - xdu = vdy - xdu$$

    - i.e. g is a function of u and y

$$\frac{\partial g}{\partial y} = v \qquad \frac{\partial g}{\partial u} = -x$$

If $f = L$ and $(x, y) = (\dot{q}, q)$

$$L(\dot{q}, q) \rightarrow \underline{g(p, q)} = L - p\dot{q}$$

This is what we need

Hamiltonian

Opposite sign from Legendre transf.

- Define Hamiltonian: $H(q,p,t) = \dot{q}_i p_i - L(q,\dot{q},t)$
  - Total derivative is

$$dH = \cancel{p_i d\dot{q}_i} + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

  - Lagrange's equations say $\dfrac{\partial L}{\partial q_i} = \dot{p}_i$

$$\Rightarrow \quad dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$

    - This must be equivalent to

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

Putting them together gives…

Hamilton's Equations

- We find $\dfrac{\partial H}{\partial p_i} = \dot{q}_i$ $\dfrac{\partial H}{\partial q_i} = -\dot{p}_i$ and $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}$
  - $2n$ equations replacing the n Lagrange's equations
  - 1st-order differential instead of 2nd-order
  - "Symmetry" between q and p is apparent
- There is nothing new – We just rearranged equations
  - First equation links momentum to velocity
    - This relation is "given" in Newtonian formalism
  - Second equation is equivalent to Newton's/Lagrange's equations of motion

Quick Example

- Particle under Hooke's law force $F = -kx$

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 \Rightarrow p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\Rightarrow H = \dot{x}p - L = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = \frac{p^2}{2m} + \frac{k}{2}x^2$$

Replace $\dot{x}$ with $\frac{p}{m}$

- Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

Usual harmonic oscillator

Energy Function

- Definition of Hamiltonian is identical to the energy function $h(q,\dot{q},t) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q,\dot{q},t)$
  - Difference is subtle: H is a function of (q, p, t)
- This equals to the total energy if
  - Lagrangian is $L = L_0(q,t) + L_1(q,t)\dot{q}_i + L_2(q,t)\dot{q}_j\dot{q}_k$
  - Constraints are time-independent
    - This makes $T = L_2(q,t)\dot{q}_j\dot{q}_k$
  - Forces are conservative
    - This makes $V = -L_0(q)$

See Lecture 4, or
Goldstein Section 2.7

Hamiltonian and Total Energy

- If the conditions make h to be total energy, we can skip calculating L and go directly to H
  - For the particle under Hooke's law force

$$H = E = T + V = \frac{p^2}{2m} + \frac{k}{2}x^2$$

- This works often, but not always
  - when the coordinate system is time-dependent
    - e.g., rotating (non-inertial) coordinate system
  - when the potential is velocity-dependent
    - e.g., particle in an EM field

Let's look at this

Particle in EM Field

- For a particle in an EM field

$$L = \frac{m}{2}\dot{x}_i^2 - q\phi + qA_i\dot{x}_i$$

We can't jump on $H = E$ because of the last term, but

$$\Rightarrow \quad p_i = m\dot{x}_i + qA_i$$

$$\Rightarrow \quad H = (m\dot{x}_i + qA_i)\dot{x}_i - L = \frac{m\dot{x}_i^2}{2} + q\phi$$

This is in fact E

  - We'd be done if we were calculating h
  - For H, we must rewrite it using $p_i = m\dot{x}_i + qA_i$

$$H(x_i, p_i) = \frac{(p_i - qA_i)^2}{2m} + q\phi$$

Particle in EM Field

$$H(x_i, p_i) = \frac{(p_i - qA_i)^2}{2m} + q\phi$$

- Hamilton's equations are

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i - qA_i}{m} \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i} = q\frac{p_j - qA_j}{m}\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

  - Are they equivalent to the usual Lorentz force?
  - Check this by eliminating p_i

$$\frac{d}{dt}(m\dot{x}_i + qA_i) = q\dot{x}_i\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

A bit of work

$$\frac{d}{dt}(mv_i) = qE_i + q(\mathbf{v} \times \mathbf{B})_i$$

Conservation of Hamiltonian

- Consider time-derivative of Hamiltonian

$$\frac{dH(q,p,t)}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$$

$$= -\cancel{\dot{p}q} + \cancel{\dot{q}p} + \frac{\partial H}{\partial t}$$

Hamiltonian is conserved if it does not depend explicitly on t

  - H may or may not be total energy
    - If it is, this means energy conservation
    - Even if it isn't, H is still a constant of motion

Cyclic Coordinates

- A cyclic coordinate does not appear in L
  - By construction, it does not appear in H either

$$H(\not{q}, p, t) = \dot{q}_i p_i - L(\not{q}, \dot{q}, t)$$

  - Hamilton's equation says

$$\dot{p} = -\frac{\partial H}{\partial q} = 0$$

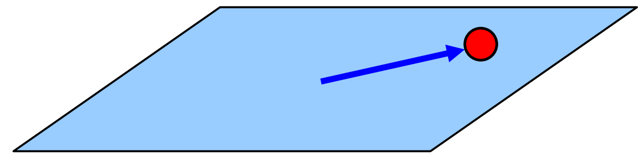
Conjugate momentum of a cyclic coordinate is conserved

  - Exactly the same as in the Lagrangian formalism

Cyclic Example

- Central force problem in 2 dimensions

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$



$$\Rightarrow \quad p_r = m\dot{r} \qquad p_\theta = mr^2\dot{\theta}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{l^2}{r^2}\right) + V(r)$$

$\Leftarrow$ θ is cyclic $\qquad p_\theta = \text{const} = l$

Hamilton's equations

$$\dot{r} = \frac{p_r}{m} \qquad \dot{p}_r = \frac{l^2}{mr^3} - \frac{\partial V(r)}{\partial r}$$

  - Cyclic variable drops off by itself

Going Relativistic

- Practical approach
  - Find a Hamiltonian that “works”
    - Does it represent the total energy?
- Purist approach
  - Construct covariant Hamiltonian formalism
    - For one particle in an EM field
- Don’t expect miracles
  - Fundamental difficulties remain the same

Practical Approach

- Start from the relativistic Lagrangian that "works"

$$L = -mc^2\sqrt{1-\beta^2} - V(\mathbf{x})$$

$$p_i = \frac{\partial L}{\partial v_i} = \frac{mv_i}{\sqrt{1-\beta^2}}$$

Did this last time

$$H = h = \sqrt{p^2c^2 + m^2c^4} + V(\mathbf{x})$$

  - It does equal to the total energy
  - Hamilton's equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i c^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{p_i}{m\gamma}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i$$

Practical Approach w/ EM Field

- Consider a particle in an EM field

$$L = -mc^2\sqrt{1-\beta^2} - q\phi(\mathbf{x}) + q(\mathbf{v}\cdot\mathbf{A})$$

  - Hamiltonian is still total energy

$$H = m\gamma c^2 + q\phi$$
$$= \sqrt{m^2\gamma^2 v^2 c^2 + m^2 c^4} + q\phi$$

Can be easily checked

  - Difference is in the momentum $p_i = m\gamma v_i + qA_i$

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi$$

Not the usual linear momentum!

Practical Approach w/ EM Field

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi$$

- Consider $H - q\phi$

  ➡ $(H - q\phi)^2 - (\mathbf{p} - q\mathbf{A})^2 c^2 = m^2 c^4$ ⇐ constant

- It means that $(H - q\phi, \mathbf{p}c - q\mathbf{A}c)$ is a 4-vector, and so is $(H, \mathbf{p}c)$

  Similar to 4-momentum $(E/c, \mathbf{p})$ of a relativistic particle

  Remember **p** here is not the linear momentum!

- This particular Hamiltonian + canonical momentum transforms as a 4-vector
  - True only for well-defined 4-potential such as EM field

Purist Approach

- Covariant Lagrangian for a free particle $\Lambda = \frac{1}{2} m u_\nu u^\nu$

$$\Rightarrow \quad p^\mu = \frac{\partial \Lambda}{\partial u_\mu} = m u^\mu \quad \Rightarrow \quad H = \frac{p_\mu p^\mu}{2m}$$

  - We know that p^0 is E/c
  - We also know that x^0 is ct…

$\Rightarrow$ **Energy is the conjugate "momentum" of time**

    - Generally true for any covariant Lagrangian
    - You know the corresponding relationship in QM

Purist Approach

- Value of Hamiltonian is

$$H = \frac{p_\mu p^\mu}{2m} = \frac{mc^2}{2}$$

This is constant!

- What is important is H's dependence on p^μ

- Hamilton's equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu}{m} \qquad \frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu} = 0$$

4-momentum conservation

- Time components are

$$\frac{d(ct)}{d\tau} = \frac{E}{mc} = \gamma c \qquad \frac{d(E/c)}{d\tau} = 0$$

Energy definition and conservation

Purist Approach w/ EM Field

- With EM field, Lagrangian becomes

$$\Lambda(x^\mu, u^\mu) = \tfrac{1}{2} m u_\mu u^\mu + q u^\mu A_\mu \quad \Rightarrow \quad p^\mu = m u^\mu + q A^\mu$$

$$\Rightarrow \quad H = \frac{m u_\mu u^\mu}{2} = \frac{(p_\mu - qA_\mu)(p^\mu - qA^\mu)}{2m}$$

  - Hamilton's equations are

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu - qA^\mu}{m} \qquad \frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu} = -\frac{(p_\nu - qA_\nu)}{m}\frac{\partial A^\nu}{\partial x_\mu}$$

  - A bit of work can turn them into

$$m\frac{du^\mu}{d\tau} = q\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right) u_\nu = K^\mu$$

← 4-force

EM Field and Hamiltonian

- In Hamiltonian formalism, EM field always modify the canonical momentum as $p_A^\mu = p_0^\mu + qA^\mu$

  With EM field → p_A^μ ; Without EM field → p_0^μ

  - A handy rule:

    Hamiltonian with EM field is given by replacing p^μ in the field-free Hamiltonian with $p^\mu - qA^\mu$

  - Often used in relativistic QM to introduce EM interaction

Summary

- Constructed Hamiltonian formalism
  - Equivalent to Lagrangian formalism
    - Simpler, but twice as many, equations
  - Hamiltonian is conserved (unless explicitly *t*-dependent)
    - Equals to total energy (unless it isn't) (duh)
  - Cyclic coordinates drops out quite easily
- A few new insights from relativistic Hamiltonians
  - Conjugate of time = energy
  - $p^\mu - qA^\mu$ rule for introducing EM interaction